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THE NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS ON 'LONG' INTERVALS

Peter A. Markowich† and Christian A. Ringhofer††

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ABSTRACT

This paper deals with the numerical solution of boundary value problems of ordinary differential equations posed on infinite intervals. The solution of these problems proceeds in two steps. The first is to cut the infinite interval at a finite, large enough point and to insert additional, so called asymptotic boundary conditions at the far (right) end; the second is to solve the resulting two point boundary value problem by a numerical method, for example a difference scheme. In this paper the Box-scheme is investigated. Numerical problems arise, because standard algorithms use too many grid points as the length of the interval increases. An 'asymptotic' a priori mesh size sequence which increases exponentially, and which therefore only employs a reasonable number of meshpoints, is developed. Investigating the conditioning of the (linearized) Box-scheme, we find that the solutions can be obtained safely by the Newton procedure when partial pivoting is employed.

AMS(MOS) Subject Classification: 34B15, 34C05, 34D05, 34A10, 34A11

Key Words: Nonlinear boundary value problems, Singular points, Asymptotic properties, Difference equations, Stability of difference equations

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SIGNIFICANCE AND EXPLANATION

This paper deals with the numerical solution of boundary value problems of ordinary differential equations posed on infinite intervals. These problems have the following form. We look for the solution of a system of ordinary differential equations which is defined on the interval $[1,\infty]$, which fulfills a certain continuity requirement at infinity and some boundary conditions at $t = 1$.

Such problems occur in laminar flow theory, in fluid dynamical stability theory, and in quantum mechanics.

For the numerical solution we proceed as follows. First we cut the infinite interval at a finite, large number $t = T$ and impose additional, so called asymptotic boundary conditions at $t = T$ in order to obtain a 'finite' two point boundary value problem. Then we solve this problem with a finite difference method. The difficulty arising frequently is that the number of mesh points has to increase rapidly as $T \to \infty$ in order to preserve a certain accuracy when standard algorithms are used.

In this paper we derive a sequence of mesh-sizes which increases exponentially. The amount of computing necessary is kept reasonably low.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

THE NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS ON 'LONG' INTERVALS

Peter A. Markowich† and Christian A. Ringhofer††

1. Introduction.

In this paper the numerical solution of boundary value problems on infinite intervals of the form

(1.1) $y' = t^{\alpha}f(t,y), \quad 1 \le t < \infty, \quad \alpha \ge 0$

(1.2) $b(y(1)) = 0$

(1.3) $y \in C([1,\infty]) :\Longleftrightarrow y \in C([1,\infty))$ and $\lim_{t\to\infty} y(t) = y(\infty)$ exists

is considered. Here $f:\mathbb{R}^{n+1} \to \mathbb{R}^n$, $b:\mathbb{R}^n \to \mathbb{R}^k$ where generally $k < n$ holds because (1.3) furnishes another set of boundary conditions. f fulfills certain continuity properties at infinity which will be defined later. We assume that the Jacobian $\frac{\partial f}{\partial y}(\infty,y(\infty))$ has no eigenvalue on the imaginary axis.

For $\alpha > -1$ Equation (1.1) has a singularity of the second kind of rank $\alpha + 1$ at $t = \infty$. But we disregard the practically unimportant case $-1 < \alpha < 0$ for the following.

Problems of this kind often occur in fluid dynamics (boundary layer theory), quantum-mechanics and electronics. For applications see Markowich (1980a,b), de Hoog and Weiss (1980a), McLeod (1969) and Schneider (1979).

For the numerical solution we proceed as follows. First the infinite interval is substituted by a finite but large interval and n-k additional, so called asymptotic boundary conditions which reflect the asymptotic behaviour of the solution y, are imposed at the right (far) endpoint T. We obtain two point boundary value problems of the form

(1.4) $x' = t^{\alpha}f(t,x), \quad 1 \le t \le T$

(1.5) $b(x(1)) = 0$

(1.6) $S(T)x(T) = \gamma(T).$

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The condition (1.6) has to be chosen such that

$$(1.7) \qquad \|y - x\|_{[1,T]} \to 0 \quad \text{as} \quad T \to \infty$$

holds and its construction is described in de Hoog and Weiss (1980b), Lentini and Keller (1980) and Markowich (1980b).

The two point boundary value problem (1.4), (1.5), (1.6) now has to be solved by an appropriate numerical method, for example by the Box-scheme which has the form

$$(1.8) \qquad \frac{x_{i+1} - x_i}{h_i} = t^{\alpha}_{i+1/2} f(t_{i+1/2}, \tfrac{1}{2}(x_{i+1} + x_i)), \qquad i = 0(1)(N-1)$$

$$(1.9) \qquad b(x_0) = 0$$

$$(1.10) \qquad S(T)x_n = \gamma(T)$$

where $t_0 = 1 < t_1 < \cdots < t_{N-1} < t_N = T$, $t_{i+1} = t_i + h_i$, $t_{i+1/2} = t_i + \frac{h_i}{2}$ holds.

It is clear that the mesh-size selection is, especially for these problems, very important since the amount of labor will be very large for long intervals and bad (too small) mesh-size choices. Even well working adaptive codes, which assume a relation of the form

$$(1.11) \qquad \max_i h_i / \min_i h_i \le \text{const}$$

and whose convergence estimates are formulated in terms of $\max_i h_i$, would emmploy too many meshpoints in order to admit a given bound for the global error. Moreover the codes which employ adaptive mesh refinement (see Lentini and Pereyra (1977)) solve first with a coarse grid in order to do local error estimation. However if T is large even a coarse grid implies a lot of computational labor and is therefore not suitable.

In this paper we use the asymptotic form of the solution of (1.1), (1.2), (1.3) in order to construct an asymptotic a priori mesh. The term asymptotic has to be understood in the following sense. In an interval $[1,\gamma]$, where γ solely depends on the 'infinite' problem either constant meshsizes or more sophisticated algorithms like equidistributing meshes (see Lentini and Pereyra (1977)) have to be used and in $[\gamma,T]$ the asymptotic mesh is employed.

It turns out that meshsizes which increase exponentially can be used since our assumptions guarantee that $y(t) \to y(\infty)$ exponentially. For the Box-scheme it will be shown that the number of grid points which is necessary in order to achieve a total accuracy $O(\varepsilon)$ (total accuracy refers to the difference between the 'infinite' solution $y(t_i)$ and the discrete approximation x_i) equals $O(\varepsilon^{-1/2})$. For this a suitable $T = T(\varepsilon)$ has been taken. Moreover it will be shown that the Newton procedure for (1.8), (1.9), (1.10) with these exponentially increasing stepsizes converges quadratically from a domain of starting values which does not shrink to ϕ as $\varepsilon \to 0$. We also show that the condition number of the (linear) Box-scheme is an $O(\varepsilon^{-1/2})$ so that the linear system can be safely solved by partial pivoting. Therefore, in the nonlinear case, the Newton procedure can be safely applied.

Of course, no fully implicit difference scheme (like the implicit Euler scheme) should be used for the integration of (1.4), (1.5), (1.6) since in general the fundamental matrix of the linearized problem (1.4) contains exponentially increasing columns which are scaled down by the boundary condition (1.6). But nevertheless this would cause instabilities during the integration when using large meshsizes. Similar instabilities can occur in the case $\alpha > 0$ when using the trapezoidal rule.

Higher order stable methods can be constructed by polynomial collocation at Gaussian points and will be analyzed in a subsequent paper.

Another way to solve problems of the kind (1.1), (1.2), (1.3) is to transform the 'infinite' problem by a transformation $t = s^{-\beta}$, $\beta > 0$ to the interval $[0,1]$ and to employ difference methods at this constant interval. Methods of this kind have been investigated by de Hoog and Weiss (1979). However this way of proceeding has the disadvantage that a singular problem (the right hand side of the equation is not defined in $s = 0$) has to be solved and therefore the obtained convergence estimates are not very strong. Another disadvantage is that many physical problems are actually posed on an infinite interval (for example in boundary layer theory) such that a 'direct' solution is desirable.

We remark that there is a close connection to singular perturbation problems since the transformation $s = \frac{t-1}{T-1}$, $\varepsilon = \frac{1}{T-1}$ takes (1.4), (1.5), (1.6) into

$$\varepsilon^{\alpha+1} z'(s) = (s+\varepsilon)^{\alpha} f(\frac{s+\varepsilon}{\varepsilon}, z(s)), \quad 0 < s < 1, \quad \alpha > 0 \tag{1.12}$$

$$b(z(0)) = 0 \tag{1.13}$$

$$s(\frac{1}{\varepsilon} + 1) z(1) = 0. \tag{1.14}$$

(Note that $\lim_{\varepsilon \to 0} f(\frac{s+\varepsilon}{\varepsilon}, z) = f(\infty, z)$).

The already developed mesh-size sequencés for singular perturbation problems cannot be applied since the linearization of the right hand side of (1.12) does not generally have a series expansion in powers of ε uniformly in $0 < s < 1$. (see Ringhofer (1981)) since for most practical problems

$$\frac{\partial f}{\partial y}(t,y) \sim \sum_{i=0}^{\infty} A_i(y) t^{-i}, \qquad t \to \infty \tag{1.15}$$

holds.

Recently Ascher and Weiss (1981) came up with a meshsize sequence for linear constant coefficient singular perturbation problems $(\alpha = 0)$ which is (as a formula) equivalent to ours, however they apply it in the layer of thickness $O(\varepsilon)$ which corresponds to the interval $[1,\gamma]$ in our long interval case. Outside the layer they use a uniform (independent of ε) mesh fine enough to approximate the solution of the reduced problem $(\varepsilon = 0)$ well. The difference comes from the fact that the solution of the singular perturbation problem decays exponentially to the solution of the reduced problem within the layer while in the infinite interval case exponential convergence holds, for $s \to 1$.

Kreiss (1975) used a similar approach to construct meshes for singular perturbation problems.

This paper is organized as follows. Chapter 2 gives a short summary of the theory of boundary value problems on infinite intervals and their 'finite' approximation, in Chapter 3 stepsize sequences are developed for scalar initial value problems, Chapters 4,5 deal with linear boundary value problems and in Chapter 6 nonlinear problems are dealt with. In Chapter 7 the results are gathered and put into algorithmic form.

2. Boundary Value Problems on Infinite Intervals and their Approximation by 'finite' Interval Problems - A Summary

We consider boundary value problems on infinite intervals of the following form

$$(2.1)\qquad y' = t^{\alpha}A(t)y + t^{\alpha}f(t),\quad 1 \le t < \infty,\quad \alpha \ge 0$$

$$(2.2)\qquad By(1) = \beta$$

$$(2.3)\qquad y \in C([1,\infty])$$

where the $n\times n$-matrix $A \in C([1,\infty])$ and $f \in C([1,\infty])$. B is a matrix whose rank is (in general) less than n since (2.3) furnishes another set of boundary conditions.

Let us first consider the case where $A(t) \equiv A$. A shall have the Jordan form J obtained by

$$(2.4)\qquad A = E\,J\,E^{-1}\,.$$

We assume that J has the block form

$$(2.5)\qquad J = \begin{array}{c} \left[\begin{array}{cc} J^{+} & \\ & J^{-} \end{array}\right] \begin{array}{l} \}r_+ \\ \}r_- \end{array} \\ \begin{array}{cc} \underbrace{}_{r_+} & \underbrace{}_{r_-} \end{array} \end{array}$$

where the $r_+ \times r_+$ matrix J^+ has only eigenvalues with positive real parts and the $r_- \times r_-$ matrix J^- has only eigenvalues with negative real parts. Imaginary eigenvalues will be excluded for the following. The diagonal projection D_+, D_- are defined by

$$(2.6)\qquad D_+ = \begin{bmatrix} I_{r_+} & \\ & 0 \end{bmatrix},$$

$$(2.7)\qquad D_- = \begin{bmatrix} 0 & \\ & I_{r_-} \end{bmatrix}.$$

The general solution of (2.1) (with $A(t) \equiv A$) and (2.3) can now be written as

$$(2.8)\qquad y(t) = E\phi(t)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix}\xi + E(HE^{-1}f)(t),\quad \xi \in C^{r_-}$$

where

$$(2.9)\qquad \phi(t) = \exp\left(\frac{J}{\alpha+1}\, t^{\alpha+1}\right)$$

is the fundamental matrix of the transformed problem

$$(2.10)\qquad u' = t^{\alpha} J u + t^{\alpha} E^{-1} f(t)$$

where $Eu = y$ holds and $(HE^{-1}f)(t)$ is a suitable particular solution of (2.10) which can be taken as

$$(2.11)\qquad (Hg)(t) = \phi(t) \int_{\infty}^{t} D_{+}\phi^{-1}(s) s^{\alpha} g(s)ds + \phi(t) \int_{\gamma}^{t} D_{-}\phi^{-1}(s) s^{\alpha} g(s)ds$$

for some $\gamma > 1$. This operator has been analyzed by de Hoog and Weiss (1980a,b) and Markowich (1980a).

H has the following properties

$$(2.12)(a)\qquad H: C([\gamma,\infty]) \to C([\gamma,\infty])$$

$$(2.12)(b)\qquad \|H\|_{[\gamma,\infty]} \le c$$

where $\|\cdot\|_{[\gamma,\infty]}$ denotes the max-norm on $[\gamma,\infty]$ resp. the associated operator-norm. The constant c is independent of γ.

Markowich (1980a) has shown that if

$$(2.13)\qquad f(t) = F(t)\exp\left(-\frac{\mu}{\alpha+1}\, t^{\alpha+1}\right), \quad F \in L_{\infty}([1,\infty]) \cap C([1,\infty])$$

holds with $\mu > \lambda_{min} > 0$ where λ_{min} is the smallest modulus of the real parts of the eigenvalues of A which are in the left half plane, then

$$(2.14)\qquad \begin{aligned} \|y(t)\| &\le \text{const.}(\|F\|_{[1,\infty]} + \|\xi\|)\left\|\phi(t)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix}\right\| \le \\ &\le \text{const.}(\|F\|_{[1,\infty]} + \|\xi\|)\, t^{(r-1)(\alpha+1)} \exp\left(-\frac{\lambda_{min}}{\alpha+1}\, t^{\alpha+1}\right) \end{aligned}$$

holds. r is the largest dimension of a Jordan block associated with an eigenvalue of A with real part $-\lambda_{min}$.

The boundary value problem (2.1), (2.2), (2.3) with $A(t) \equiv A$ is uniquely soluble for all $\beta \in \mathbb{R}^{r_-}$, $f \in C([1,\infty])$ if and only if the $r_- \times r_-$ matrix

$$(2.15)\qquad BE\phi(1)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix} \text{ is nonsingular.}$$

Here B is assumed to be a $r_- \times n$-matrix. So the continuity requirement (2.3) furnishes r_+ linearly independent boundary conditions.

(2.15) implies that $\|\xi\| \le \text{const.}\|\beta\|$ holds.

The variable coefficient case $A(t) \not\equiv A$ is treated by a perturbation approach (see de Hoog and Weiss (1980a,b) and Markowich (1980a,b)). $A(\infty)$ now plays the role of A. We assume that $A(\infty)$ has the Jordan form J given by (2.5). Then we can show (see de Hoog and Weiss (1980a,b)) that

(2.16) $$y(t) = E\psi_-(t)\xi + E\psi(E^{-1}f)(t), \quad \xi \in \mathbb{C}^{r_-}$$

where $\psi_-(t)$ is an $n \times r_-$-matrix defined by

(2.17) $$\psi_-(\cdot) = (I - H(E^{-1}A(\cdot)E - J))^{-1}\phi(\cdot)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix} \in C([\gamma,\infty])$$

for γ sufficiently large. For $t \in [1,\gamma]$ $\psi_-(t)$ can be continuously extended. $E\psi(E^{-1}f)$ is a suitable particular solution of (2.1). The boundary value problem (2.1), (2.2), (2.3) is uniquely soluble for every $\beta \in \mathbb{R}^{r_-}$, $f \in C([1,\infty])$ if and only if the $r_- \times r_-$-matrix

(2.18) $BE\psi_-(1)$ is nonsingular.

Markowich (1980a) has proven the estimate

(2.19) $$\|y(t)\| \le \text{const } (\|F\|_{[1,\infty]} + \|\beta\|)\exp\left(-\frac{(\lambda_{min}-\delta)}{\alpha+1}\, t^{\alpha+1}\right)$$

for $t \ge \bar{t}$. $\delta > 0$ can be chosen sufficiently small if $\bar{t}$ is large.

Now we briefly consider nonlinear problems of the form (1.1), (1.2), (1.3).

From (1.1), (1.3) we conclude that

(2.20) $$f(\infty,y(\infty)) = 0$$

has to hold. We assume that the roots $y(\infty)$ of (2.20) are isolated and take one particular root $y^*(\infty)$ for the following. Moreover $f(t,y^*(\infty))$ shall fulfill (2.13). Defining

$$C_\kappa(\bar{t},a) = \{(t,y) \in \mathbb{R}^{n+1} \mid t \ge \bar{t}\ ,\ \|y - y^*(\infty)\| \le \kappa\}$$

we assume that f, $f_y \in C_{lip}(C_\kappa(1,y^*(\infty)))$ for a sufficiently large κ. We also assume that the boundary value problems (1.1), (1.2),. (1.3) has an isolated solution, i.e. the linearized problem is nonsingular.

Now let J be the Jordan form of $f_y(\infty,y^*(\infty))$ obtained by

$$f_y(\infty,y^*(\infty)) = EJE^{-1}$$

and let J fulfill (2.5) such that D_+, D_-, are defined as in (2.6), (2.7). Then λ_{min}

is defined for J as above and $\phi(t)$, $\psi_-(t)$ are as in (2.9), (2.17) with $f_y(t,y^*(\infty))$ substituted for $A(t)$. Markowich (1980a) showed that

$$(2.21) \qquad \|y(t) - y^*(\infty)\| \le \text{const.}\|\psi_-(t)\| \le \text{const.}\ \exp\left(-\frac{(\lambda_{min}-\delta)}{\alpha+1}\, t^{\alpha+1}\right).$$

The isolatedness of y now implies that

$$(2.22) \qquad \frac{\partial b}{\partial y}(y(1))\psi_-(1)$$

is nonsingular. More information on the analysis of these problems can be found in the above cited references and in Lentini and Keller (1980).

We want to approximate the 'infinite' problems (2.1), (2.2), (2.3) by 'finite' two point boundary value problems of the form

$$(2.23) \qquad x' = t^{\alpha}A(t)x + t^{\alpha}f(t), \quad 1 \le t \le T\ , \quad T \gg 1$$

$$(2.24) \qquad Bx(1) = \beta$$

$$(2.25) \qquad S(T)x(t) = \gamma(T) \quad .$$

Since (2.24) is a boundary condition of rank r_- we assume the $S(T)$ is a $r_+ \times n$-matrix. The question that arises immediately is how to construct an asymptotic boundary condition $S(T)$ such that

$$(2.26) \qquad \|y - x\|_{[1,T]} \to 0 \quad \text{as} \quad T \to \infty$$

and the order of convergence should be as fast as possible.

A complete theory of this kind can be found in de Hoog and Weiss (1980a) and Markowich (1980b) and therefore we only give excerpts which will be needed in the sequel. The basic idea is that the boundary condition (2.25) has to scale down all solution components of (2.23), which do not decay exponentially.

We assume that (2.18) holds. A possible choice is

$$(2.27) \qquad S \equiv S(T) = [I_{r_+}, 0]E^{-1}, \ \gamma(T) = 0 \ .$$

It has been shown in the above cited references that this boundary condition implies convergence in the sense of (2.26) and that for general $\gamma(T)$

$$(2.28) \qquad \|y - x\|_{[1,T]} \le \text{const}\|Sy(T) - \gamma(T)\|$$

holds. In general the admissibility conditions for a boundary condition $S(T)$ are

$$(2.29) \qquad \|S(T)\| \le \text{const.}, \qquad T \to \infty$$

$$(2.30) \qquad \left\| \left(S(T)E \begin{bmatrix} I_{r_+} \\ 0 \end{bmatrix} \right)^{-1} \right\| \le \text{const.}, \quad T \to \infty$$

then (2.28) holds for the unique solution x of the 'finite' problem if T is sufficiently large. $\gamma(T) \equiv 0$ is a natural choice for linear problems.

If $f(t)$ fulfills (2.13) an estimate for the order of convergence is given by the right hand side of (2.19). Moreover it has been shown that the choice (2.27) is optimal in the sense that the actual order of convergence exceeds (2.19) for homogenous problems.

The condition (2.25) with S fulfilling (2.29), (2.30) and $\gamma(T) = S(T)y^*(\infty)$ can also be used for nonlinear problems of the form (1.1), (1.2), (1.3) if the above stated assumptions on $f(t,y)$ and the solution y hold. (2.28) still holds for nonlinear problems.

3. The Scalar Case.

In this chapter we treat the simplest case, namely scalar initial value problems. The aim is to construct step-size sequences for the Box-scheme such that the global error is less than a prescribed accuracy regardless of the length of the interval of integration. These step-size sequences will be used for the general boundary value problem case.

We consider

$$(3.1) \qquad y' = -\lambda t^{\alpha} y + t^{\alpha} f(t) , \qquad 1 < t < \infty , \qquad \alpha > 0$$

$$(3.2) \qquad y(1) = \bar{y}$$

where $\lambda = \lambda_1 + i\lambda_2$ may vary in a compact subset Ω of $\{z \in \mathbb{C} | \mathrm{Re}\, z > 0\}$.

The Box- or centered Euler scheme for (3.1), (3.2) has the form

$$(3.3) \qquad \frac{y_{i+1} - y_i}{h_i} = - \frac{\lambda}{2} t^{\alpha}_{i+1/2}(y_{i+1} + y_i) + t^{\alpha}_{i+1/2} f_{i+1/2} , \ i > 0 \ ; \ y_0 = \bar{y}$$

where for $h_i > 0$

$$(3.4) \quad (a) \quad t_0 = 1,\ t_{i+1} = t_i + h_i ,\ t_{i+1/2} = t_i + \frac{h_i}{2} ,\ i > 0; \quad (b) \quad f_{i+1/2} = f(t_{i+1/2})$$

holds.

We define

$$(3.5)(a) \qquad Y_{n,m}(\lambda,h) = \prod_{\ell=n}^{m} \frac{1 - \frac{\lambda}{2} h_{\ell} t^{\alpha}_{\ell+1/2}}{1 + \frac{\lambda}{2} h_{\ell} t^{\alpha}_{\ell+1/2}} , \quad n < m ; \quad Y_{n+1,n} = 1 , \quad n > -1$$

and for a sequence of complex numbers $z = (z_{j+1/2})_{j=I}^{i-1}$

$$(3.5)(b) \qquad (H_{-}(\lambda,t_I,h)z)_i = \sum_{\ell=I}^{i-1} \frac{h_{\ell} t^{\alpha}_{\ell+1/2} z_{\ell+1/2}}{1 + \frac{\lambda}{2} h_{\ell} t^{\alpha}_{\ell+1/2}} Y_{\ell+1,i-1}(\lambda,h), \ i > I+1$$

and $(H_{-}(\lambda,t_I,h)z)_I = 0$ where $h = (h_j)_{j=0}^{i-1}$ is the sequence of step-sizes.

Using these definitions the solution of (3.4) can be written as

$$(3.6) \qquad y_i = Y_{0,i-1}(\lambda,h)\bar{y} + (H_{-}(\lambda,t_0,h)f)_i$$

where $f = (f_{j+1/2})_{j=0}^{i-1}$ has been set.

The local discretization error $t^{\alpha}_{i+1/2}\ell_{i+1/2}$ of the difference scheme (3.3) is defined as

$$(3.7) \qquad t^{\alpha}_{i+1/2}\ell_{i+1/2} = \frac{y(t_{i+1})-y(t_i)}{h_i} + \frac{\lambda}{2} t^{\alpha}_{i+1/2}(y(t_{i+1}) + y(t_i)) - t^{\alpha}_{i+1/2} f_{i+1/2}, \quad i \geq 0$$

where y is the solution of (3.1), (3.2).

The global error

$$(3.8) \qquad e_i = y(t_i) - y_i$$

then satisfies the difference equation

$$(3.9) \qquad \frac{e_{i+1}-e_i}{h_i} = -\frac{\lambda}{2} t^{\alpha}_{i+1/2}(e_{i+1} + e_i) + t^{\alpha}_{i+1/2}\ell_{i+1/2}, \quad i \geq 0, \quad e_0 = 0$$

and therefore has the solution

$$(3.10) \qquad e_i = (H_-(\lambda,t_0,h)\ell)_i$$

with $\ell = (\ell_{j+1/2})^{i-1}_{j=0}$.

In order to estimate the right hand side of (3.10) we need the following

Lemma 3.1. Let x_κ for $\kappa = n,n+1,\cdots,m$ be complex numbers with $\operatorname{Re} x_\kappa > 0$ and

$\frac{\operatorname{Im} x_\kappa}{\operatorname{Re} x_\kappa} \leq x = \text{const}$. Then setting $\prod_{\ell=i+1}^{i} a_i = 0$ for $i \geq 1$

$$(3.11) \qquad \sum_{\kappa=n}^{m} \frac{|x_\kappa|}{|1+x_\kappa|^2} \prod_{\ell=\kappa+1}^{m} \frac{|1-x_\ell|}{|1+x_\ell|} \leq 1/2\sqrt{1+x^2}$$

$$(3.12) \qquad \sum_{\kappa=n}^{m} \frac{|x_\kappa|}{|1+x_\kappa|^2} \prod_{\ell=n}^{\kappa-1} \frac{|1-x_\ell|}{|1+x_\ell|} \leq 1/2\sqrt{1+x^2}$$

holds.

Proof: An easy calculation gives

$$\frac{|x_\kappa|}{|1+x_\kappa|^2} \leq 1/2\sqrt{1+x^2}\left(1 - \frac{|1-x_\kappa|}{|1-x_\kappa|}\right).$$

Substitution into the right hand sides of (3.11), (3.12) yields telescoping sums.

Application of Lemma 3.1 immediately yields

Lemma 3.2. Let $f = (f_{\ell+1/2})^{i-1}_{\ell=I}$. Then for every sequence $h = (h_\ell)^{i-1}_{\ell=I}$ with $h_\ell > 0$

$$(3.13)\qquad |(H_-(\lambda,t_I,h)f)_i| \le \text{const.} \max_{\ell=I(1)(i-1)} [|f_{\ell+1/2}|(1+\frac{|\lambda|}{2}h_\ell t^\alpha_{\ell+1/2})]$$

holds for $i > I$ uniformly for $\lambda \in \Omega$.

Proof:

$$|(H_-(\lambda,t_I,h)f)_i| \le$$

$$\le \text{const} \max_{\ell=I(1)(i-1)} (|f_{\ell+1/2}|(1+\frac{|\lambda|}{2}h_\ell t^\alpha_{\ell+1/2}))\sum_{\ell=I}^{i-1}\frac{h_\ell t^\alpha_{\ell+1/2}}{|1+1/2\, h_\ell t^\alpha_{\ell+1/2}|^2} Y_{\ell+1,i-1}(\lambda,h)$$

and application of (3.11) yields (3.13).

We get from (3.10)

$$(3.14)\qquad |e_i| \le \text{const.} \max_{\ell=0(1)(i-1)} [|\ell_{\ell+1/2}|(1+\frac{|\lambda|}{2}h_\ell t^\alpha_{\ell+1/2})].$$

For the following we assume that

$$(3.15)\qquad f(t) = F(t)\exp(-\frac{\mu}{\alpha+1}t^{\alpha+1}), \quad \mu > \lambda_1$$

holds where $F,F',F'' \in C([1,\infty)) \cap L_\infty([1,\infty])$.

A straightforward calculation gives

$$(3.16)\qquad |\ell_{\ell+1/2}| \le \text{const.}\, h_\ell^2[\frac{1}{t^\alpha_{\ell+1/2}}\|y'''\|_{[t_\ell,t_{\ell+1}]} + |\lambda|\,\|y''\|_{[t_\ell,t_{\ell-1}]}].$$

Markowich (1980a) shows that (3.14) implies

$$|y(t)| \le c_1(\lambda)(\|F\|_{[1,\infty]} + |\bar{y}|)\exp(-\frac{\lambda_1}{\alpha+1}t^{\alpha+1})$$

where $c_1(\lambda)$ is bounded for $\lambda \in \Omega$. Differentiating (3.1) and using (3.16) yields

$$(3.17)\qquad |\ell_{\ell+1/2}| \le c_2(\lambda)(\sum_{\kappa=0}^{2}\|F^{(\kappa)}\|_{[1,\infty]} + |\bar{y}|)t_\ell^{2\alpha}\exp(-\frac{\lambda_1}{\alpha+1}t_\ell^{\alpha+1}), \qquad t_\ell \ge t_{(1)}(\lambda)$$

where the function $t^{3\alpha}\exp(-\frac{\lambda_1}{\alpha+1}t^{\alpha+1})$ takes its maximum over $[1,\infty]$ at $t_{(1)}(\lambda)$.

$c_2(\lambda)$ and $t_{(1)}(\lambda)$ are bounded for $\lambda \in \Omega$. We get

$$(3.18)\qquad \begin{aligned} |e_i| &\le c[\sum_{\kappa=0}^{2}\|F^{(\kappa)}\|_{[1,\infty]} + |\bar{y}|)(t_{(1)}^{\alpha+1}\max_{\ell=0(1)I} h_\ell^2 + \\ &+ \max_{\ell=(I+1)(1)(i-1)} [h_\ell^2 t_\ell^{2\alpha}\exp(-\frac{\lambda_1}{\alpha+1}t_\ell^{\alpha+1})(1+h_\ell t^\alpha_{\ell+1/2})] .\end{aligned}$$

Here $t_I \le \bar{t}_{(1)} \le t_{I+1}$ holds, where $\bar{t}_{(1)} = \max_{\lambda\in\Omega} t_{(1)}(\lambda)$ and c is independent of $\lambda \in \Omega$. (3.18) gives

$$|e_i| \le 2c\Big(\sum_{\kappa=0}^{2} \|F^{(\kappa)}\|_{[1,\infty]} + |\bar{y}|\Big)\Big(\bar{t}_{(1)}^{\alpha+1} \max_{\ell=0(1)I} h_\ell^2 +$$

(3.19)

$$+ \max_{\ell=(I+1)(1)(i-1)} \Big[t_\ell^{2\alpha} \exp\Big(-\frac{\lambda_1}{\alpha+1} t_\ell^{\alpha+1}\Big) \max(h_\ell^2, h_\ell^3 t_\ell^\alpha, h_\ell^{\alpha+3})\Big] .$$

Now we require that for some $0 < \varepsilon \le \varepsilon_0$

(3.20) $$|e_i| \le 4c\Big[\sum_{\kappa=0}^{2} \|F^{(\kappa)}\|_{[1,\infty]} + |\bar{y}|\Big]\varepsilon , \qquad i > 0$$

holds. This is fulfilled if we choose

(3.21)(a) $$\max_{\ell=0(1)I} h_\ell \le \sqrt{\frac{\varepsilon}{t_{(1)}^{\alpha+1}}} = \bar{h}(\lambda_1, \varepsilon, t_\ell), \quad t_\ell \le t_{(1)}$$

and for $\ell > I$

(3.21)(b) $$h_\ell \le \bar{h}(\lambda_1,\varepsilon,t_\ell) = \begin{cases} \sqrt{\varepsilon}\, t_\ell^{-\alpha} \exp\Big(\frac{\lambda_1}{2(\alpha+1)} t_\ell^{\alpha+1}\Big), & t_{(1)} \le t_\ell \le \Big(\frac{\alpha+1}{\lambda_1} \ln \frac{1}{\varepsilon}\Big)^{\frac{1}{\alpha+1}} = t_{(2)}(\varepsilon) \\ \sqrt[3]{\varepsilon}\, t_\ell^{-\alpha} \exp\Big(\frac{\lambda_1}{3(\alpha+1)} t_\ell^{\alpha+1}\Big), & t_{(2)}(\varepsilon) \le t_\ell \le t_{(3)}(\varepsilon) \\ \sqrt[\alpha+3]{\varepsilon}\, t_\ell^{-\frac{2\alpha}{\alpha+3}} \exp\Big(\frac{\lambda_1}{(\alpha+3)(\alpha+1)} t_\ell^{\alpha+1}\Big), & t_\ell \ge t_{(3)}(\varepsilon) \end{cases}$$

where $t_{(3)}(\varepsilon) \ge t_{(2)}(\varepsilon)$ is the root of $\frac{1}{\sqrt[3]{\varepsilon}} = t_{(3)}^{-(\alpha+1)} \exp\Big(\frac{\lambda_1}{3(\alpha+1)} t_{(3)}^{\alpha+1}\Big)$. (Note that $t_{(2)}(\varepsilon) = t_{(3)}(\varepsilon)$ for $\alpha = 0$.)

This defines upper bounds for the stepsizes at a given point t_ℓ depending on ε, the bound for the global error, and on $\lambda_1 = \mathrm{Re}\lambda$. These bounds are independent of the length of the integration interval and increase (in $t = t_\ell$) exponentially.

We now compute the number of steps $N(T,\varepsilon)$ which is necessary for integration on the interval $[1,T]$ if $h_\ell = \bar{h}(\lambda_1,\varepsilon,t_\ell)$ is chosen. Therefore we write

(3.22) $$N(T,\varepsilon) = \sum_{i\in I_{(0)}} 1 + \sum_{i\in I_{(1)}} 1 + \sum_{i\in I_{(2)}} 1 + \sum_{i\in I_{(3)}} 1$$

where $I_{(n)} = \{i \in \mathbb{N}_0 | t_i \in (t_{(n)}, t_{(n+1)}]\}$ where $t_{(0)} = t_0 = 1$ and $t_{(4)} = T$ has been set. Obviously

(3.23) $$\sum_{i\in I_{(0)}} 1 \le \frac{(t_{(1)}-1)\sqrt{t_{(1)}^{\alpha+1}}}{\sqrt{\varepsilon}}$$

$$(3.24)\qquad \sum_{i\in I_{(1)}} 1 = \sum_{i\in I_{(1)}} \frac{h_i}{h_{i+1}} \frac{h_{i+1}}{h_i} \leq \max_{i\in I_{(1)}} \left(\frac{h_{i+1}}{h_i}\right) \int_{t_{(1)}}^{t_{(2)}} \frac{dt}{\bar{h}(\lambda,\varepsilon,t)} \leq c(\lambda)\frac{1}{\sqrt{\varepsilon}}$$

where $c(\lambda)$ is bounded in Ω since $\max\limits_{i\in I_{(1)}} \frac{h_{i+1}}{h_i} \leq e^{\frac{\lambda_1}{2}}$ holds. The same estimate is easily obtained for $\sum\limits_{i\in I_{(2)}} 1$.

For $t_{(3)}(\varepsilon) \leq t_i \leq T$ we get

$$(3.25)\qquad h_{i+1} \geq 2^{-\frac{2\alpha}{\alpha+3}} \exp\left(\frac{\lambda_1}{(\alpha+3)(\alpha+1)} (2^{\alpha+1}-1)t_{(3)}^{\alpha+1}\right)h_i .$$

Since $t_{(3)}^{\alpha+1} \geq \frac{\alpha+1}{\lambda_1} \ln \frac{1}{\varepsilon}$ holds we derive

$$(3.26)\qquad h_{i+1} \geq 2^{-\frac{2\alpha}{\alpha+3}} \varepsilon^{-\frac{(2^{\alpha+1}-1)}{\alpha+3}} h_i \geq 2^{-\frac{2\alpha}{\alpha+3}} \varepsilon^{-\frac{1}{3}} h_i$$

and get a bound $K = \sum\limits_{i\in I_{(3)}} 1$ from

$$(3.27)\qquad T \geq \sum_{i=j_1}^{j_1+k-1} h_i$$

where $I_{(3)} = \{j_1,\cdots,j_1 + K-1\}$ has been set. Further we use

$$h_{j_1} \geq t_{(3)}(\varepsilon) \geq \left(\frac{\alpha+1}{\lambda_1}\right)^{\frac{1}{\alpha+1}} \left(\ln \frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}}$$

such that

$$(3.28)\qquad h_{j_1+\ell} \geq \left[2^{-\frac{2\alpha}{\alpha+3}} \varepsilon^{-\frac{1}{3}}\right]^{\ell} \left(\frac{\alpha+1}{\lambda_1}\right)^{\frac{1}{\alpha+1}} \left(\ln \frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}} .$$

From (3.27), (3.28) we conclude

$$(3.29) \qquad \sum_{i \in I_{(3)}} 1 \leq c(\lambda)\left(1 + \frac{\ln T}{\ln \frac{1}{\varepsilon}}\right).$$

Altogether this gives

$$(3.30) \qquad N(T,\varepsilon) \leq c(\lambda)\left(\frac{1}{\sqrt{\varepsilon}} + \frac{\ln T}{\ln \frac{1}{\varepsilon}}\right).$$

A constant stepsize algorithm would need

$$(3.31) \qquad N_{const}(T,\varepsilon) \sim \frac{T}{\min(\sqrt{\varepsilon}, \sqrt[3]{\frac{\varepsilon}{T^{\alpha}}})}$$

steps because

$$\max_i |(H_-(\lambda,t_0,h_{const})\ell)_i| \leq const(1 + \frac{|\lambda|}{2} hT^{\alpha})\max_i |\ell_{i+1/2}|, \; t_{i+1} \leq T$$

holds where $h_{const} = (h)_{i=0}^{\infty}$ is a sequence of constant stepsizes.

Therefore the stepsize sequence $\bar{h}$ given by (3.21)(a),(b) is very efficient and the reason for this is that no condition like $h_{max}/h_{min} \leq const$ or even $h_{i+1}/h_i \leq const$ is required. We remark that $\bar{h}$ equidistributes the local error.

The problem (3.1), (3.2) can be regarded as a model for the decaying solution components of boundary value problems on 'long' intervals and now we look at the increasing components, which can be modelled by

$$(3.32) \qquad z' = \omega t^{\alpha} z + t^{\alpha} f(t), \quad 1 \leq t \leq T, \quad \alpha > 0$$

$$(3.33) \qquad z(T) = \bar{z}$$

where $\omega = \omega_1 + i\omega_2 \in \Omega$ and Ω is again a compact subset of $\{z \in \mathbb{C} | \mathrm{Re}\, z > 0\}$. We again use the Box-scheme to approximate (3.32), (3.33)

$$(3.34) \qquad \frac{z_{i+1} - z_i}{h_i} = \frac{\omega}{2} t^{\alpha}_{i+1/2}(z_{i+1}+z_i) + t^{\alpha}_{i+1/2} f_{i+1/2}, \quad i \geq 0; \quad z_N = \bar{z}$$

where

$$(3.35) \qquad t_0 = 1 < t_1 = t_0+h_0 < \cdots < t_{N-1} = t_{N-2}+h_{n-2} < t_N = t_{N-1}+h_{N-1} = T$$

holds.

The solution of (3.34) is given by

$$z_i = Y_{i,N-1}(\omega,h)\bar{z} + (H_+(\omega,t_N,h)f)_i \tag{3.36}$$

where

$$(H_+(\omega,t_N,h)f)_i = -\sum_{\ell=i}^{N-1} \frac{h_\ell t^\alpha_{\ell+1/2} f_{\ell+1/2}}{1+\frac{\omega}{2}h_\ell t^\alpha_{\ell+1/2}} Y_{i,\ell-1}(\omega,h), \quad i \leq N-1 \tag{3.37}$$

and $(H_+(\omega,t_N,h)f)_N = 0$.

Since the increasing components are scaled down by the asymptotic boundary conditions at $t = T$ we disregard the convergence of the z_i to $z(t_i)$ but we prove a stability estimate analogously to Lemma 3.2.

<u>Lemma 3.3.</u> Let $f = (f_{i+1/2})_{\ell=1}^{N-1}$. Then for every sequence $h = (h_\ell)_{\ell=1}^{N-1}$

$$|(H_+(\omega,t_N,h)f)_i| \leq \text{const.} \max_{\ell=i(1)(N-1)} [|f_{\ell+1/2}|(1+\frac{|\omega|}{2}h_\ell t^\alpha_{\ell+1/2})] \tag{3.38}$$

holds for $i \leq N$ uniformly for $\omega \in \Omega$.

<u>Proof.</u>

$$|(H_+(\omega,t_N,h)f)_i| \leq$$

$$\leq \max_{\ell=i(1)(N-1)} (|f_{\ell+1/2}|(1+\frac{|\omega|}{2}h_\ell t^\alpha_{\ell+1/2}))\sum_{\ell=i}^{N-1} \frac{h_\ell t^\alpha_{\ell+1/2}}{|1+\frac{\omega}{2}h_\ell t^\alpha_{\ell+1/2}|^2} |Y_{i,\ell-1}(\omega,h)|.$$

Application of Lemma 3.1, (3.12), yields:

$$|(H_+(\omega,t_N,h)f)_i| \leq c(\omega) \max_{\ell=i(1)(N-1)} |f_{\ell+1/2}|(1+\frac{|\omega|}{2}h_\ell t^\alpha_{\ell+1/2}).$$

Finally we prove

<u>Lemma 3.4.</u> Assume that $t_i \leq t_j \leq t_{(2)}(\varepsilon)$ and that $h_\ell \leq \bar{h}(\lambda_1,\varepsilon,t_\ell)$ for $\lambda_1 > 0$

Then $$|Y_{i,j-1}(\omega,h)| \leq \exp(-c(t_j - t_i))$$

and $c = c(\omega)$ is bounded on Ω .

Proof. Let $z = z_1 + iz_2$, $z_1 > 0$. Then

$$\left|\frac{1-z}{1+z}\right|^2 < 1 - \frac{4z_1}{|1+z|^2} < \exp\left(-4 \frac{z_1}{|1+z|^2}\right)$$

holds. This estimate has been used in de Hoog and Weiss (1979]. Therefore for $\omega = \omega_1 + i\omega_2$

$$|Y_{i,j-1}(\omega,h)| < \exp\left(-2\omega_1 \sum_{\ell=i}^{j-1} \frac{h_\ell t^\alpha_{\ell+1/2}}{|1 + \frac{\omega}{2} h_\ell t^\alpha_{\ell+1/2}|^2}\right) .$$

Since $|1 + \frac{\omega}{2} h_\ell t^\alpha_{\ell+1/2}|^2 < c(\omega)$ for $t_i < t_j < t_2(\varepsilon)$ holds we get

$$|Y_{i,j-1}(\omega,h)| < \exp(- \frac{2\omega_1}{c(\omega)} \sum_{\ell=i}^{j-1} h_\ell t^\alpha_{\ell+1/2}) < \exp(-c \sum_{\ell=i}^{j-1} h_\ell) < \exp(-c(t_j - t_i)).$$

4. The Case $A(t) \equiv A$.

We consider

$$(4.1) \qquad x' = t^{\alpha}Ax + t^{\alpha}f(t), \quad 1 \le t \le T, \quad \alpha > 0$$

$$(4.2) \qquad Bx(1) = \beta$$

$$(4.3) \qquad S(T)x(T) = \gamma(T)$$

where A fulfills (2.4), (2.5), B is an $r_- \times n$-matrix, $\beta \in \mathbb{R}^{r_-}$ and f fulfills (2.13) with $\mu > \lambda_{min}$, the $r_+ \times n$-matrix $S(T)$ fulfills (2.29), (2.30). This simple case shall be considered as a model for problems where A depends on t.

The Box-scheme has the form

$$(4.4) \qquad \frac{x_{i+1} - x_i}{h_i} = \frac{A}{2} t^{\alpha}_{i+1/2}(x_{i+1} + x_i) + t^{\alpha}_{i+1/2} f_{i+1/2} \quad i = 0(1)(N-1)$$

$$(4.5) \qquad Bx_0 = \beta$$

$$(4.6) \qquad S(T)x_N = \gamma(T)$$

where the partition $\Delta = \{t_0, t_1, \cdots, t_{N-1}, t_N\}$ fulfills (3.35) and $h = (h_i)_{i=0}^{N-1}$, $h_i > 0$. A fulfills (2.4). Now we employ the transformation

$$(4.7) \qquad x_i = Eu_i$$

and get

$$(4.8) \qquad \frac{u_{i+1} - u_i}{h_i} = \frac{J}{2} t^{\alpha}_{i+1/2}(u_{i+1} + u_i) + t^{\alpha}_{i+1/2} E^{-1} f_{i+1/2}, \quad i = 0(1)(N-1)$$

$$(4.9) \qquad BEu_0 = \beta$$

$$(4.10) \qquad S(T)Eu_N = \gamma(T) \quad .$$

We want to derive an existence and stability theorem for (4.8), (4.9), (4.10). As de Hoog and Weiss (1980a) did for the continuous case we split u_i into

$$(4.11) \qquad (a) \quad u_i = \begin{pmatrix} u_i^+ \\ u_i^- \end{pmatrix} \begin{matrix} \}r_+ \\ \}r_- \end{matrix}, \qquad (b) \quad E^{-1}f_{i+1/2} = \begin{pmatrix} (E^{-1}f_{i+1/2})^+ \\ (E^{-1}f_{i+1/2})^- \end{pmatrix} \begin{matrix} \}r_+ \\ \}r_- \end{matrix}$$

and get employing (3.6) for u_i^- resp (3.36) for u_i^+

$$(4.12)\qquad u_i = \begin{bmatrix} Y^{r_+}_{i,N-1}(J^+,h) \\ 0 \end{bmatrix}\xi_+ + \begin{bmatrix} 0 \\ Y^{r_-}_{0,i-1}(-J^-,h)e^{J^-} \end{bmatrix}\xi_- + (H(J,t_0,t_N,h)E^{-1}f)_i$$

where for any $k \times k$ matrix P whose eigenvalues have positive real part

$$(4.13)\quad Y^k_{n,m}(P,h) = \prod_{\ell=n}^{m} (I_k + \frac{P}{2} h_\ell t^\alpha_{\ell+1/2})^{-1}(I_k - \frac{P}{2} h_\ell t^\alpha_{\ell+1/2}), \quad m > n, \quad Y_{n+1,n}(P,h) = I,\ n > -1$$

holds and the operator H is defined as

$$(4.14)(a)\qquad (H(J,t_0,t_N,h)z)_i = \begin{pmatrix} (H_+(J^+,t_N,h)z^+)_i \\ (H_-(-J^-,t_0,h)z^-)_i \end{pmatrix}, \quad z = \begin{pmatrix} z^+ \\ z^- \end{pmatrix}$$

and

$$(4.14)(b)\qquad (H_+(J^+,t_N,h)z^+)_i = -\sum_{\ell=i}^{N-1} h_\ell Y^{r_+}_{i,\ell-1}(J^+,h)(I + \frac{J^+}{2} h_\ell t^\alpha_{\ell+1/2})^{-1} t^\alpha_{\ell+1/2} z^+_{\ell+1/2}$$

$$(4.14)(c)\qquad (H_-(-J,t_I,h)z^-)_i = \sum_{\ell=I}^{i-1} h_\ell Y^{r_-}_{\ell+1,i-1}(-J^-,h)(I - \frac{J^-}{2} h_\ell t^\alpha_{\ell+1/2})^{-1} t^\alpha_{\ell+1/2} z^-_{\ell+1/2}$$

where $z^+ = (z^+_{i+1/2})^{N-1}_{i=0}$, $z^- = (z^-_{i+1/2})^{N-1}_{i=0}$ and $z^+_{i+1/2} \in C^{r_+}$, $z^-_{i+1/2} \in C^{r_-}$ has been set.

Here $(H_+(J^+,t_N,h)z^+)_N = 0$ and $(H_-(-J^-,t_I,h)z^-)_I = 0$ hold. These definitions make sense because $(I + \tau J^+)^{-1}$, $(I - \tau J^-)^{-1}$ exist for $\tau > 0$.

In order to get bounds for the defined difference operators we use the following well known representation of a matrix function

$$(4.15)\qquad \varphi(P) = \frac{1}{2\pi i}\int_{\Gamma_P} \varphi(\lambda)(\lambda I - P)^{-1} d\lambda$$

where the contour Γ_P encloses all eigenvalues of P. φ is assumed to be analytic.

If all eigenvalues of P have positive real parts we get

$$(4.16)\qquad Y^k_{n,m}(P,h) = \frac{1}{2\pi i}\int_{\Gamma_P} Y_{n,m}(\lambda,h)(\lambda I_k - P)^{-1} d\lambda$$

where $Y_{n,m}$ is defined in (3.4) and

$$(4.17)\qquad (H(J,t_0,t_N,h)z)_i = \frac{1}{2\pi i}\begin{pmatrix} \int_{\Gamma_+} (H_+(\omega,t_N,h)(\omega I - J^+)^{-1} z)_i d\omega \\ \int_{\Gamma_-} (H_-(-\lambda,t_0,h)(\lambda I - J^-)^{-1} z^-)_i d\lambda \end{pmatrix}$$

where $\Gamma_+ \subset \{z \in C | \text{Re}z > 0\}$, $\Gamma_- \subset \{z \in C | \text{Re}z < 0\}$ holds. Since

$$(4.18) \qquad \max_{\omega \in \Gamma_+} \|(\omega I - J^+)^{-1}\|, \quad \max_{\lambda \in \Gamma_-} \|(\lambda I - J^-)^{-1}\| < \text{const.}$$

holds, the estimates given in Chapter 3 can be used because they were formulated uniformly for $-\lambda, \omega$ in compact subsets of the left half plane.

By evaluating (4.12) at the boundaries ($i = 0$ resp $i = N$) we get the block system

$$(4.19) \qquad \begin{bmatrix} BE\begin{bmatrix} Y^{r_+}_{0,N-1}(J^+,h) \\ 0 \end{bmatrix} & BE\begin{bmatrix} 0 \\ e^{J^-} \end{bmatrix} \\ S(T)E\begin{bmatrix} I \\ 0 \end{bmatrix} & S(T)E\begin{bmatrix} 0 \\ Y^{r_-}_{0,N-1}(-J^-,h) \end{bmatrix} \end{bmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} =$$

$$= \begin{pmatrix} \beta - BE(H(J,t_0,t_N,h)E^{-1}f)_0 \\ \gamma(T) - S(T)E(H(J,t_0,t_N,h)E^{-1}f)_N \end{pmatrix}.$$

We assume that (2.1), (2.2), (2.3) with $A(T) \equiv A$ has a unique solution for all f's $\in C([1,\infty])$, $\beta \in R^{r_-}$. Therefore (2.15) has to hold which implies that $BE\begin{bmatrix} 0 \\ e^{J^-} \end{bmatrix}$ is non-singular. From (2.30) we conclude that $S(T)E\begin{bmatrix} I \\ 0 \end{bmatrix}$ has a bounded inverse. From (4.16) we conclude that

$$(4.20) \qquad \|Y^{r_+}_{0,N-1}(J^+,h)\| < \text{const.} \max_{\omega \in \Gamma_+} |Y_{0,N-1}(\omega,h)| < \text{const.}$$

and

$$(4.21) \qquad \|Y^{r_-}_{0,N-1}(-J^-,h)\| < \text{const.} \max_{\lambda \in \Gamma_-} |Y_{0,N-1}(-\lambda,h)| .$$

Now let $\bar{\lambda}$ be the eigenvalue of J^- which is nearest to the imaginary axis, such that $\text{Re}\bar{\lambda} = -\lambda_{min}$ and take Γ_- such that for some small $\delta > 0$

$$(4.22) \qquad \text{dist}(\Gamma_-,\bar{\lambda}) = \delta \quad \text{and} \quad \text{dist}(\Gamma_-, \{z \in C | \text{Re}z = 0\}) = \lambda_{min} - \delta$$

holds. We now choose $h = (h_i)_{i=0}^{N-1}$ such that $h_i < \bar{h}(\lambda_{min}-\delta,\varepsilon,t_i)$ defined in (3.21).

Then

(4.23) $$\max_{\lambda\in\Gamma_-} |Y_{0,N-1}(-\lambda,h)| \le \text{const}(\varepsilon + \exp(-\frac{(\lambda_{min}-\delta)}{\alpha+1}T^{\alpha+1}))$$

holds. Therefore, for ε sufficiently small and T sufficiently large, $\|Y^{r_-}_{0,N-1}(-J^-,h)\|$ can be made sufficiently small such that the block system (4.19) has a unique solution ξ_+, ξ_- and

(4.24) $$\max_{i=0(1)N} \|x_i\| \le \text{const}\ (\|\beta\| + \max_{i=0(1)N} \|(H(J,t_0,t_N,h)E^{-1}f)_i\| + \|\gamma(T)\|)$$

holds where const. is independent of ε and T.

Now we set $\gamma(T) \equiv 0$.

The local discretization error $t^\alpha_{i+1/2}\ell_{i+1/2}$ is again defined as

(4.25) $$t^\alpha_{i+1/2}\ell_{i+1/2} = \frac{x(t_{i+1}) - x(t_i)}{h_i} - \frac{A}{2}t^\alpha_{i+1/2}(x_{i+1}+x_i) - t^\alpha_{i+1/2}f_{i+1/2}, \quad i = 0(1)(N-1)$$

such that the global discretization error

(4.26) $$e_i = x(t_i) - x_i$$

fulfills the discrete boundary value problem

(4.27) $$\frac{e_{i+1}-e_i}{h_i} = \frac{A}{2}t^\alpha_{i+1/2}(e_{i+1}+e_i) + t^\alpha_{i+1/2}\ell_{i+1/2}, \quad i = 0(1)(N-1)$$

(4.28) $$Be_0 = 0$$

(4.29) $$S(T)e_N = 0 .$$

From (4.24) we get

(4.30) $$\max_{i=0(1)N} \|e_i\| \le \text{const.} \max_{i=0(1)N} \|(H(J,t_0,t_N,h)\ell)_i\|$$

where $\ell = (\ell_{i+1/2})^{N-1}_{i=0}$ has been set.

(4.17) and the Lemmas (3.2), (3.3) yield

(4.31) $$\|(H(J,t_0,t_N,h)\ell)_i\| \le \text{const.} \max_{\ell=0(1)(N-1)} [\|\ell_{\ell+1/2}\|(1 + \frac{c}{2}h_\ell t^\alpha_{\ell+1/2})]$$

for some $c > 0$. As in (3.16) we get

(4.32) $$\|\ell_{\ell+1/2}\| \le \text{const}\ h^2_\ell[\frac{1}{t^\alpha_{\ell+1/2}}\|x'''\|_{[t_\ell,t_{\ell+1}]} + \|x''\|_{[t_\ell,t_{\ell+1}]})$$

assuming that (2.13) holds with $F,F',F'' \in C([1,\infty)] \cap L_\infty([1,\infty])$. Since

$$\|x(t)\| \le \|y(t)\| + \|x(t) - y(t)\| \le \|y(t)\| + \text{const}\|S(T)y(T)\|$$

holds where y is the (unique) solution of the 'infinite' problem (2.1), (2.2), (2.3)

with $A(T) \equiv A$ we get

$$(4.33) \qquad \|x(t)\| \le \text{const.}\Big(\Big(\sum_{\kappa=0}^{2} \|F^{(\kappa)}\|_{[1,\infty]} + \|\beta\|\Big)\exp\Big(-\frac{(\lambda_{min}-\delta)}{\alpha+1} t^{\alpha+1}\Big) + \|S(T)y(T)\|\Big) .$$

This follows from (2.14), (2.28). By using the differential equation (4.1) we derive

$$(4.34) \qquad \|\ell_{\ell+1/2}\| \le \text{const.}h_\ell^2\Big(t_\ell^{2\alpha}\exp\Big(-\frac{(\lambda_{min}-\delta)}{\alpha+1} t_\ell^{\alpha+1}\Big) + T^2\|S(T)y(T)\|\Big)$$

for $t_\ell > \bar{t} = t_{(1)}(\lambda_{min}-\delta)$ which is defined in (3.17). We assume that, given an $\varepsilon > 0$, $h_\ell \le \bar{h}(\lambda_{min}-\delta,\varepsilon,t_\ell)$ defined in (3.21). This yields

$$(4.35) \qquad \max_{i=0(1)N} \|e_i\| \le \text{const.}\ (\varepsilon + T^{5+\alpha}\|S(T)y(T)\|) .$$

Using the estimate (2.14) for $y(T)$ gives a bound for the total error

$$(4.36) \qquad \max_{i=0(1)N} \|y(t_i)-x_i\| \le \text{const.}\Big(\varepsilon + \exp\Big(-\frac{(\lambda_{min}-\delta)}{\alpha+1} T^{\alpha+1}\Big)\Big) .$$

Again y solves the 'infinite' problem and x_i are the solutions of the Box-scheme for the 'finite' problem on $[1,T]$. δ is sufficiently small.

In order to achieve a total occurance of $O(\varepsilon)$ the choice

$$(4.37) \qquad T = T(\varepsilon) = \Big(\frac{\alpha+1}{(\lambda_{min}-\delta)} \ln\frac{1}{\varepsilon}\Big)^{\frac{1}{\alpha+1}}$$

is sufficient.

Let us summarize the results.

We solve (4.4), (4.5), (4.6) with a stepsize sequence h_i fulfilling

$$(4.38) \qquad h_\ell \le \sqrt{\frac{\varepsilon}{\bar{t}^{\alpha+1}}} \ , \quad t_\ell \le \bar{t}$$

$$(4.39) \qquad h_\ell \le \sqrt{\varepsilon}\ t_\ell^{-\alpha}\exp\Big(\frac{(\lambda_{min}-\delta)}{\alpha+1} t_\ell^{\alpha+1}\Big),\ t_\ell > \bar{t}$$

on the interval $\Big[1, \Big(\frac{\alpha+1}{(\lambda_{min}-\delta)} \ln\frac{1}{\varepsilon}\Big)^{\frac{1}{\alpha+1}}\Big]$ and get the error estimate

$$(4.40) \qquad \max_i \|y(t_i) - x_i\| = O(\varepsilon) \ , \quad \varepsilon \to 0 .$$

The number of necessary steps $N(\varepsilon)$ when equality holds in (4.38), (4.39) is given by (3.23), (3.24)

(4.41) $$N(\varepsilon) \sim c\frac{1}{\sqrt{\varepsilon}}, \quad \varepsilon \to 0$$

which is comparable with the number of steps a constant stepsize algorithm would need for the integration of a constant interval problem in order to achieve an $0(\varepsilon)$ accuracy.

Of course the second term in the error estimate (4.36) can be reduced by adding gridpoints $t_\ell > \left(\frac{\alpha+1}{(\lambda_{min}-\delta} \ell n \frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}}$ and by forming stepsizes h_ℓ according to (3.21)(b) (second and third branch of $\bar{h}$). An estimate for the number of gridpoints in the case $h_i = \bar{h}(\lambda_{min}-\delta,\varepsilon,t_\ell)$ is given by (3.30).

5. The General Linear Case.

The 'infinite' problem (2.1), (2.2), (2.3) shall be approximated by the finite problem (2.23), (2.24), (2.25). The asymptotic boundary condition $S(T)$ shall fulfill the regularity conditions (2.29), (2.30).

The Box-scheme for the approximating problem is

$$\frac{x_{i+1}-x_i}{h_i} = \tfrac{1}{2}\, t^{\alpha}_{i+\frac{1}{2}} A(t_{i+\frac{1}{2}})(x_{i+1}+x_i) + t_{i+\frac{1}{2}} f_{i+\frac{1}{2}}, \quad i = 0(1)(N-1) \tag{5.1}$$

$$Bx_0 = \beta \tag{5.2}$$

$$S(T)x_N = \gamma(T) \quad . \tag{5.3}$$

For the $n \times n$ matrix $A \in C([1,\infty])$ is assumed to hold and $A(\infty) = \lim_{t\to\infty} A(t)$ has the Jordan for J obtained by

$$A(\infty) = EJE^{-1} \tag{5.4}$$

and J has the block structure (2.5). Again we set

$$x_i = Eu_i \tag{5.5}$$

and define

$$G(t) = E^{-1}A(t)E - J \quad ; \quad G(t) \to 0 \qquad t \to \infty . \tag{5.6}$$

Again we want to derive an existence and stability theorem, but now we use a perturbation approach and a contraction argument.

We rewrite (5.1), (5.2), (5.3)

$$\frac{u_{i+1}-u_i}{h_i} = \frac{J}{2}\, t^{\alpha}_{i+\frac{1}{2}} (u_{i+1}+u_i) + \tfrac{1}{2}\, t^{\alpha}_{i+\frac{1}{2}} G(t_{i+\frac{1}{2}})(u_{i+1}+u_i) + t^{\alpha}_{i+\frac{1}{2}} E^{-1} f_{i+\frac{1}{2}} \tag{5.7}$$

$$i = 0(1)(N-1)$$

$$BEu_0 = \beta \tag{5.8}$$

$$S(T)Eu_N = \gamma(T) \quad . \tag{5.9}$$

According to Chapter 4 the general solution of (5.7) can be written as

$$u_i = \begin{bmatrix} Y^{r_+}_{i,N-1}(J^+,h) \\ 0 \end{bmatrix} \xi_+ + \begin{bmatrix} 0 \\ Y^{r_-}(-J^-,h)e^{J^-} \end{bmatrix} \xi_- + \tag{5.10}$$

$$+ (H(J,t_I,t_N,h)\tilde{G}u)_i + (H(J,t_I,t_N,h)E^{-1}f)_i$$

(5.11) $$(\tilde{G}u)_{\ell+1/2} = \tfrac{1}{2} G(t_{\ell+1/2})(u_{\ell+1}+u_\ell), \quad \tilde{G}u = ((\tilde{G}u)_{\ell+1/2})_{\ell=I}^{N-1}$$

has been set. $\xi_+ \in C^{r_+}$ and $\xi_- \in C^{r_-}$ hold.

From (4.17) and the Lemmas 3.2 and 3.3 we get

(5.12) $$\max_{\ell=I(1)N} \|(H(J,t_I,t_N,h)\tilde{G}u)_\ell\| \le$$
$$\le \text{const.}\|G\|_{[t_I,t_N]} \max_{\ell=I(1)(N-1)} (1 + \tfrac{c}{2} h_\ell t^\alpha_{\ell+1/2}) \max_{\ell=I(1)N} \|u_\ell\| .$$

We choose $T = t_N = \left(\frac{\alpha+1}{(\lambda_{min}-\delta)} \ln \frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}}$ and $h_\ell \le \bar{h}(\lambda_{min}-\delta,\varepsilon,t_\ell)$ defined in (3.21). $\lambda_{min} > 0$ is again the modulus of the real part of that eigenvalue of $A(\infty)$ which is closest to the imaginary axis of all eigenvalues of $A(\infty)$ with negative real part and $\delta > 0$ is sufficiently small.

Since $\bar{h}(\lambda_1,\varepsilon,t_{\ell+1}) \ge \bar{h}(\lambda_1,\varepsilon,t_\ell)$ we derive from (3.21) that $\bar{h}(\lambda_{min} - \delta,\varepsilon,t_{N-1}) \le T^{-\alpha}$ such that $1 + \frac{c}{2} h_\ell t^\alpha_{\ell+1/2} \le$ const holds. We define the operator

(5.13) $$\hat{H}(h,t_I,t_N) : C^{n(N-I)} \to C^{n(N-I+1)}$$

such that for $x \in^{n(N-I)}$, $x = (x_{\ell+1/2})_{\ell=I}^{N-1}$

(5.14) $$\hat{H}(h,t_I,t_N)x = \begin{pmatrix} (H(J,t_I,t_N,h)x)_I \\ \vdots \\ (H(J,t_I,t_N,h)x)_N \end{pmatrix}$$

holds. From (5.12) we get

(5.15) $$\|\hat{H}(h,t_I,t_N)\tilde{G}\| \le \text{const.} \|G\|_{[t_I,t_N]}$$

where $\|\cdot\|$ denotes the max-norm for vectors in the respective C^j resp. the associated matrix norm. Therefore the operator

(5.16) $$I - \hat{H}(h,t_I,t_N)\tilde{G} : C^{n(N-I+1)} \to C^{n(N-I+1)}$$

is invertible for $t_I \le t_N$ sufficiently large. We define

$$(5.17)\qquad (a)\ Y^{+}_{I,N}(J^{+},h) = \begin{bmatrix} \begin{bmatrix} Y^{r_+}_{I,N-1}(J^{+},h) \\ 0 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} Y^{r_+}_{N,N-1}(J^{+},h) \\ 0 \end{bmatrix} \end{bmatrix} \quad , \ (b)\ Y^{-}_{I,N}(-J^{-},h) = \begin{bmatrix} \begin{bmatrix} 0 \\ Y^{r_-}_{0,I-1}(-J^{-},h)e^{J^{-}} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} 0 \\ Y^{r_-}_{0,N-1}(-J^{-},h)e^{J_-} \end{bmatrix} \end{bmatrix}$$

such that

$$\begin{pmatrix} u_I \\ u_{I+1} \\ \vdots \\ u_{N-1} \\ u_N \end{pmatrix} = (I-\hat{H}(h,t_I,t_N)\tilde{G})^{-1} Y^{+}_{I,N}(J^{+},h)\xi_+ + (I-\hat{H}(h,t_I,t_N)\tilde{G})^{-1} Y^{-}_{I,N}(-J^{-},h)\xi_- + $$
$$+ (I-\hat{H}(h,t_I,t_N)\tilde{G})^{-1}\hat{H}\hat{E}^{-1}f$$

holds. In order to obtain $u_0,u_1,\cdots,u_{I-1}$ the difference equation (5.7) has to be solved backwards with given u_I.

Therefore $h_0,\cdots,h_{I-1}$ have to be chosen such that $(I - \frac{h_\ell}{2} t^{\alpha}_{\ell+1/2} A(t_{\ell+1/2}))^{-1}$ exist for $\ell = 0(1)(I-1)$. From this and (5.18) we get

$$(5.19)\qquad u_i = Z^{+}_i \xi_+ + Z^{-}_i \xi_- + z_i(f) , \qquad i = 0(1)N$$

where Z^{+}_i is a $n \times r_+$, Z^{-}_i is a $n \times r_-$ matrix and $z_i(f) \in C^n$.

The block system

$$(5.20)\qquad \begin{bmatrix} BEZ^{+}_0 & BEZ^{-}_0 \\ S(T)EZ^{+}_N & S(T)EZ^{-}_N \end{bmatrix} \begin{pmatrix} \xi_+ \\ \xi_- \end{pmatrix} = \begin{pmatrix} \beta - BEZ_0(f) \\ \gamma(T) - S(T)EZ_N(f) \end{pmatrix}$$

results by evaluating at the boundaries $i = 0$ resp $i = N$ and by using (5.8), (5.9). We set

$$(5.21)\qquad (a)\ Z^{+}_{I,N} = \begin{pmatrix} Z^{+}_I \\ \vdots \\ Z^{+}_N \end{pmatrix} \quad , \ (b)\ Z^{-}_{I,N} = \begin{pmatrix} Z^{-}_I \\ \vdots \\ Z^{-}_N \end{pmatrix} .$$

From (5.18) we get

$$(5.22)\qquad Z^{+}_{I,N} = \sum_{\ell=0}^{\infty} (\hat{H}(h,t_I,t_N)\tilde{G})^{i} Y^{+}_{I,N}(J^{+},h)$$

and since $\|\hat{H}(h,t_I,t_N)\| \le$ const as $\varepsilon \to 0$

$$\|Z_{I,N} - Y^{+}_{I,N}(J^{+},h)\|$$

(5.23)
$$< \text{const.} \max_{\ell=I(1)(N-1)} \left\|G(t_{\ell+\frac{1}{2}})\left(\begin{bmatrix} Y^{r_+}_{\ell,N-1}(J^{+},h) \\ 0 \end{bmatrix} + \begin{bmatrix} Y^{r_+}_{\ell+1,N-1}(J^{+},h) \\ 0 \end{bmatrix}\right)\right\| <$$

$$< \text{const.} \max_{\ell=I(1)(N-1)} \left(\|G(t_{\ell+\frac{1}{2}})\|e^{-c(T-t_{\ell+1})}\right).$$

This follows from (4.12) and Lemma 3.4. It is easy to check that the right hand side of (5.23) converges to zero as $T \to \infty$ $(\varepsilon \to 0)$. Therefore

(5.24)
$$Z^{+}_{N} = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] + o(1), \qquad \varepsilon \to 0$$

follows and

$$S(T)EZ^{+}_{N} = S(T)E\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] + o(1)$$

is nonsingular for ε sufficiently small because of (2.30) and the matrix in the (1.1) position of (5.20) is bounded as $\varepsilon \to 0$.

We conclude from (5.18)

(2.25)
$$Z^{-}_{I,N} = \hat{H}(h,t_I,t_N)\tilde{G}Z^{-}_{I,N} + Y^{-}_{I,N}(-J^{-},h).$$

In Chapter 2, (2.16) it was noted that the general solution of the homogenous problem

$$u' = t^{\alpha}(J+G(t))u$$

$$u \in C([1,\infty])$$

can be written as

(5.26)
$$\psi_{-}(t) = ((I - HG)^{-1}\phi(\cdot)\left[\begin{smallmatrix} 0 \\ I_{r_-} \end{smallmatrix}\right])(t), \qquad t > \gamma$$

where the solution operator H is defined in (2.11) and $\phi(t)$ is as in (2.9). We get

(5.27)
$$\psi_{-}(t) = (HG\psi_{-})(t) + \phi(t)\left[\begin{smallmatrix} 0 \\ I_{r_-} \end{smallmatrix}\right], \qquad t > \gamma$$

and define the vector $\psi^{-}_{I,N}$ for $\gamma = t_I$

(5.28)
$$\psi^{-}_{I,N} = \begin{pmatrix} \psi_{-}(t_I) \\ \vdots \\ \psi_{-}(t_N) \end{pmatrix} = \underbrace{\begin{pmatrix} (HG\psi_{-})(t_I) \\ \vdots \\ (HG\psi_{-})(t_N) \end{pmatrix}}_{(HG\psi_{-})_{I,N}} + \underbrace{\begin{pmatrix} \phi(t_I)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix} \\ \vdots \\ \phi(t_N)\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix} \end{pmatrix}}_{\phi^{-}_{I,N}}$$

Subtracting from (5.25) we get

$$(5.29)\qquad Z^-_{I,N} - \psi^-_{I,N} = \hat{H}(h,t_I,t_N)\tilde{G}Z^-_{I,N} - (HG\psi_-)_{I,N} + Y^-_{I,N}(-J^-,h) - \phi^-_{I,N}$$

and

$$(5.30)\qquad \begin{aligned} Z^-_{I,N} - \psi^-_{I,N} = \hat{H}(h,t_I,t_N)\tilde{G}(Z_{I,N} - \psi^-_{I,N}) - [(HG\psi_-)_{I,N} - \hat{H}(h,t_I,t_N)\tilde{G}\psi^-_{I,N}] \\ + Y^-_{I,N}(-J^-,h) - \phi^-_{I,N} \end{aligned}$$

follows. This gives

$$(5.31)\qquad \begin{aligned} Z^-_{I,N} - \psi^-_{I,N} = (I - \hat{H}(h,t_I,t_N)\tilde{G})^{-1}(Y^-_{I,N}(-J^-,h) - \phi^-_{I,N}) - \\ - (I - \hat{H}(h,t_I,t_N)\tilde{G})^{-1}((HG\psi_-)_{I,N} - \hat{H}(h,t_I,t_N)\tilde{G}\psi^-_{I,N}). \end{aligned}$$

From Chapter 4 we conclude that

$$(5.32)\qquad \|Y^-_{I,N}(-J^-,h) - \phi^-_{I,N}\| = O(\varepsilon)\ , \qquad \varepsilon \to 0$$

and since $(HG\psi_-)(t)$ is the solution of the problem

$$v' = t^{\alpha}Jv + t^{\alpha}G(t)\psi_-(t)$$
$$\begin{bmatrix} 0 \\ I_{r_-} \end{bmatrix} v(t_I) = 0$$
$$v \in C([t_I,\infty])$$

we get assuming that $A \in C([1,\infty])$; $A', A'' \in C([1,\infty)) \cap L_\infty([1,\infty])$

$$(5.33)\qquad \|(HG\psi_-)_{I,N} - \hat{H}(h,t_I,t_N)\tilde{G}\psi^-_{I,N}\| = O(\varepsilon)\ , \qquad \varepsilon \to 0$$

and therefore

$$(5.34)\qquad \|Z^-_{I,N} - \psi^-_{I,N}\| = O(\varepsilon)\ , \qquad \varepsilon \to 0$$

follows. By continuation (5.33) holds with $I = 0$.

We now assume that the 'infinite' problem (2.1), (2.2), (2.3) is uniquely soluble for every $\beta \in R^{r_-}$, $f \in C([1,\infty])$ such that (2.18) holds. Then the matrix in the (1.2) position of (5.20) is nonsingular for ε sufficiently small and its inverse if bounded as $\varepsilon \to 0$. The matrix in the (2.2) position is bounded and therefore the block system (5.20) is uniquely soluble for ε sufficiently small and we get the stability estimate

$$(5.35)\qquad \max_{i=0(1)N} \|x_i\| \le \text{const}(\|\beta\| + \|f\|_{[1,T]} + \|\gamma(T)\|).$$

By Proceeding as in Chapter 4 we get

Theorem 5.1. Assume that $A \in C([1,\infty])$; A', $A'' \in C([1,\infty)) \cap L_\infty([1,\infty])$ and that f fulfills (2.13) with F, F', $F'' \in C([1,\infty)) \cap L_\infty([1,\infty])$. Let for some ε sufficiently small

(5.36) $$T(\varepsilon) = t_N = \left(\frac{\alpha+1}{(\lambda_{min}-\delta)} \ln \frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}}$$

hold with some fixed small δ and assume that

(5.37) $$h_i < c_0\sqrt{\varepsilon} \quad \text{for} \quad t_i < \gamma , \quad c_0 > 0$$

(5.38) $$h_i < \sqrt{\varepsilon}\, t_i^{-\alpha} \exp\left(\frac{(\lambda_{min}-\delta)}{\alpha+1} t_i^{\alpha+1}\right) , \quad \gamma < t_i < T(\varepsilon)$$

hold for some fixed γ sufficiently large. Then if the matrix (2.18) is nonsingular the Box-scheme (5.1), (5.2), (5.3) is uniquely soluble for ε sufficiently small and

(5.39) $$\max_{i=0(1)N} \|x_i - y(t_i)\| = O(\varepsilon)$$

holds for $\gamma(T) \equiv 0$.

If equality holds in (5.37), (5.38) the number of steps $N = N(\varepsilon)$ fulfills

(5.40) $$N(\varepsilon) \sim c_1 \frac{1}{\sqrt{\varepsilon}} , \quad \varepsilon \to 0 .$$

The condition number of a nonsingular A is defined by

(5.41) $$\varkappa(A) = \|A\|\|A^{-1}\| .$$

Then the condition number of the difference operator L_h (given by (5.4), (5.5), (5.6) fulfills the estimate

(5.42) $$\varkappa(L_h) < \text{const} \frac{1}{\sqrt{\varepsilon}} \sim \text{const.}\ N(\varepsilon)$$

if equality holds in (5.37), (5.38).

This holds since $\|L_h\| < \text{const.} \frac{1}{\sqrt{\varepsilon}}$ and because of the stability estimate (5.35).

(5.42) is a very moderate condition number and therefore (5.4), (5.5), (5.6) can be safely solved (by partial pivoting using SOLVEBLOCK(de Boor and Weiss (1980)).

6. Nonlinear Problems.

We consider the 'infinite' problem

(6.1) $$y' = t^{\alpha} f(t,y), \quad 1 \le t < \infty, \quad \alpha > 0$$

(6.2) $$b(y(1)) = 0$$

(6.3) $$y \in C([1,\infty])$$

and the Box-scheme

(6.4) $$\frac{x_{i+1}-x_i}{h_i} = t^{\alpha}_{i+1/2} f(t_{i+1/2}, 1/2(x_{i+1}+x_i)), \quad i = 0(1)(N-1)$$

(6.5) $$b(x_0) = 0$$

(6.6) $$S(T)x_N = S(T)y^*(\infty)$$

where $T = t_N$ holds. The asymptotic boundary condition $S(T)$ is considered with regard to Chapter 2.

As mentioned in Chapter 2

(6.7) $$f(\infty, y(\infty)) = 0$$

has to hold. We now assume that there is an isolated zero $y^*(\infty)$ and that $f_y(\infty, y^*(\infty))$ has the Jordan form J obtained by

(6.8) $$f_y(\infty, y^*(\infty)) = EJE^{-1}$$

where J fulfills (2.5). Moreover we assume

(6.9) $$f \in C^2(C_{\underset{r}{x}}(1, y^*(\infty)); \ f(t, y^*(\infty)) = O(e^{-\frac{\mu}{\alpha+1} t^{\alpha+1}}), \ \mu > \lambda_{min}$$

(6.10) $$b: \mathbb{R}^n \to \mathbb{R}^{-} ; \ b, b_y \text{ are locally Lipschitz continuous in } \mathbb{R}^n$$

and that the problem (6.1), (6.2), (6.3) has an isolated solution $y^*(t) \to y^*(\infty)$ as $t \to \infty$. The isolatedness means that the linearized problem

$$z' = f_y(t, y^*(t))z$$

$$b_y(y^*(1))z(1) = 0$$

$$z \in C([1,\infty])$$

has only the trivial solution $z \equiv 0$. Then we conclude from de Hoog and Weiss (1980a) that the approximating problem

(6.11) $$x' = t^{\alpha} f(t,x)$$

(6.12) $$b(x(1)) = 0$$

(6.13) $$S(T)x(T) = S(T)y^*(\infty)$$

with (2.29), (2.30) are locally (around $y^*(t)$) uniquely soluble for T sufficiently large and

$$\|x - y\|_{[1,T]} < \text{const.}\|S(T)(y(T) - y^*(\infty))\| \tag{6.14}$$

holds. An estimate for $S(T)(y(T) - y^*(\infty))$ can be obtained from (2.21). Possible choices for $S(T)$ are discussed in Lentini and Keller (1980).

Now we choose $t_N = T = \left(\frac{\alpha+1}{(\lambda_{min}-\delta)} \ln \frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}}$ for ε sufficiently small and apply the nonlinear stability theory given in Keller (1975) with ε as a parameter. The result then follows from the stability estimate (5.35) for linear problems and we merely state it in

<u>Theorem 6.1.</u> Under the given assumption the Box-scheme is convergent for $\varepsilon \to 0$ to the locally unique solution of (6.4), (6.5), (6.6) if stepsize sequence h_ℓ fulfilling

$$h_\ell < c_0\sqrt{\varepsilon}\,, \quad t_\ell < \gamma \tag{6.15}$$

$$h_\ell < \sqrt{\varepsilon}\, t_\ell^{-\alpha} \exp\left(\frac{(\lambda_{min}-\delta)}{\alpha+1} t_\ell^{\alpha+1}\right) \qquad \gamma < t_\ell < T(\varepsilon) \tag{6.16}$$

is chosen. The estimate

$$\max_{i=0(1)N} \|y(t_i) - x_i\| = O(\varepsilon)\,, \qquad \varepsilon \to 0 \tag{6.17}$$

holds. The Newton procedure for (6.4), (6.5), (6.6) is quadratically convergent for starting values in a sphere

$$K = \left\{x \in \mathbb{C}^{n(N+1)} \,\middle|\, \left\|x - \begin{pmatrix} y(t_0) \\ \vdots \\ y(t_N) \end{pmatrix}\right\| < \xi\right\}$$

where ξ is independent of ε.

7. Algorithm

A difficulty that might arise is that $T(\varepsilon) = \left(\frac{\alpha+1}{(\lambda_{min} - \delta)} \ln \frac{1}{\varepsilon}\right)^{\frac{1}{\alpha+1}}$ is rather small for reasonable ε's $(10^{-4} < \varepsilon < 10^{-8})$. For example assume $\alpha = 1$, $\lambda_{min} \approx 2$, $\varepsilon = 10^{-8}$. Then $T \approx 4$.

If $y(t)$ has not 'reached' its asymptotic state at $T \approx 4$ no good approximation by the finite interval problem posed on $[1,4]$ can be expected. However a significant increase of T (such that $y(T)$ is reasonably close to $y(\infty)$) which corresponds to an enourmous decrease of ε would imply a large increase in the amount of labor since $N(\varepsilon) = 0(\varepsilon^{-1/2})$.

A reasonable way to overcome this difficulty is to set

(7.1) $$T(\varepsilon) = \left(\frac{\alpha+1}{(\lambda_{min} - \delta)} \ln \frac{\kappa}{\varepsilon}\right)^{\frac{1}{\alpha+1}}$$

where

(7.2) $$\kappa = \max_{t\in[1,\infty]} \left|(y(t) - y(\infty))\exp\left(\frac{\lambda_{min} - \delta}{\alpha+1} t^{\alpha+1}\right)\right|$$

has been set.

Then

(7.3) $$\|y(T(\varepsilon)) - y(\infty)\| < \varepsilon \quad \text{for all } \varepsilon > 0 .$$

The meshsize sequence (6.15), (6.16) can still be used on the whole interval $[1,T(\varepsilon)]$. The error estimate (6.17) takes the form

(7.4) $$\max_{i=0(1)N} \|x_i - y(t_i)\| = 0(\kappa\varepsilon), \quad \varepsilon \to 0$$

and $N(\varepsilon)$ would still be an $0(\varepsilon^{-1/2})$.

In the case that the function f, which sets up the differential equation, is independent of t, the choice

(7.5) $$S(T) \equiv S = [I_{r_+}, 0]E^{-1}$$

(and $\gamma(T) \equiv \gamma = Sy(\infty)$) implies

(7.6) $$\|x - y\|_{[1,T]} < \text{const } \|y(T) - y(\infty)\|^2$$

(see Lentini and Keller (1980)). Therefore we can choose

(7.7) $$T(\varepsilon) = \frac{1}{2}\left(\frac{\alpha+1}{\lambda_{min} - \delta}\ \ln \frac{\kappa}{\varepsilon}\right)^{\frac{1}{\alpha+1}}$$

and (7.4) still holds.

A way to construct an adaptive code based on the given theory is the following.

At first choose a T so large that $y(T)$ is reasonably close to $y(\infty)$. This might be done by using physical information on the solution of the infinite problem. Then choose a coarse grid on $[1,T]$ such that the meshsizes increase exponentially as $t \to T$. The solution of the equations (6.4), (6.5), (6.6) gives an initial guess $(x_0,\cdots,x_N)$. Now we calculate

(7.8) $$\tilde{\kappa} = \max_{i=0(1)N} |(x_i - y(\infty))\exp(\frac{\lambda_{min} - \delta}{\alpha+1}\ t_i^{\alpha+1})| \ .$$

Here δ should be an input parameter. Then we set $\gamma = t_I$ (for the meaning of γ see Theorem 6.1) such that

(7.9) $$\max_{\ell=I(1)N} |x_\ell - y(\infty)| < \frac{\tilde{\kappa}}{M}\ ;\ |x_j - y(\infty)| > \frac{\tilde{\kappa}}{M} \quad \text{for some } j < I$$

where M is also an input parameter.

For $i = 1(1)(I-1)$ we calculate the first order term of the local error ℓ_i using the x_i's as Lentini and Pereyra (1977) did. On $[1,t_I]$ we define an equidistributing mesh proceeding as in the just cited reference given an $\varepsilon > 0$. On $[t_I, \tilde{T}(\varepsilon)]$ where $\tilde{T}(\varepsilon)$ is given by (7.1) resp. (7.7) with $\tilde{\kappa}$ set for κ , we choose the meshsizes h_i as

$$h_i = \sqrt{\varepsilon}\ t_\ell^{-\alpha} \exp\left(\frac{\lambda_{min} - \delta}{\alpha+1}\ t_i^{\alpha+1}\right)\ , \quad i = I(1)(N-2)$$

$$h_{N-1} = \tilde{T}(\varepsilon) - t_{N-1}$$

and solve (6.4), (6.5), (6.6) on $[1,\tilde{T}(\varepsilon)]$ using the constructed mesh.

A standard error estimation algorithm then checks whether a given accuracy has been achieved. If yes then the algorithm stops, if not we calculate $\tilde{\kappa}_{NEW}$, γ_{NEW} from the just obtained solutions of the Box-scheme and calculate a new grid with $\varepsilon_{NEW} = \varepsilon/2$ separately on $[1,\gamma_{NEW}]$, $[\gamma_{NEW}\ ,\ \tilde{T}(\varepsilon_{NEW})]$ as in the first iteration.

This interative procedure stops as soon as the required accuracy is obtained.

Numerical experiments will be reported in a subsequent paper solely concerned with computational aspects of 'infinite' boundary value problems.

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This paper deals with the numerical solution of boundary value problems of ordinary differential equations posed on infinite intervals. The solution of these problems proceeds in two steps. The first is to cut the infinite interval at a finite, large enough point and to insert additional, so called asymptotic boundary conditions at the far (right) end; the second is to solve the resulting two point boundary value problem by a numerical method, for example a difference scheme. In this paper the Box-scheme is investigated. Numerical problems arise, because standard algorithms use too many grid points as the length of the interval increases. An 'asymptotic'

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20. Abstract (continued)

a priori mesh size sequence which increases exponentially, and which therefore only employs a reasonable number of meshpoints, is developed. Investigating the conditioning of the (linearized) Box-scheme we find that the solutions can be obtained safely by the Newton procedure when partial pivoting is employed.